Generating Functions

Stochastics

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Definition

If X is a discrete random variable, then the *probability generating function* or simply *generating function* of X is

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Example. If X is the value of a roll with a fair 6-sided die, then

$$G(z) = \frac{1}{6}z^1 + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6.$$

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If $X \sim PGEO(p)$, then

$$G(z) = \sum_{k=0}^{\infty} p(1-p)^k z^k = p\sum_{k=0}^{\infty}((1-p)z)^k = \frac{p}{1-(1-p)z}.$$

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Basic properties of the generating function:

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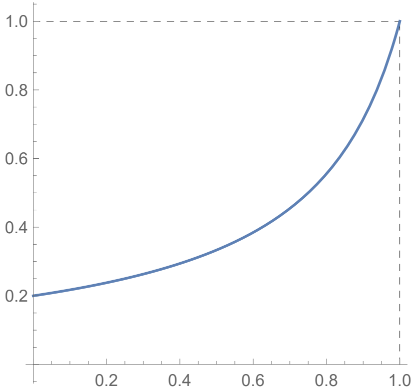
  (keeping in mind that $0^0 = 1$),
- the sum is convergent for $z \in [0, 1]$,
- $G(z)$ is analytic on $[0, 1)$,
- $G(z)$ is increasing and convex on $[0, 1]$.

Example

Example. The generating function of PGEO(4/5) is

$$G(z) = \frac{1-4/5}{1-4z/5} = \frac{1}{5-4z}.$$



Mean and variance

Lemma

- $\mathbb{E}(X) = G'(1)$,
- $\mathbb{D}(X) = \sqrt{G''(1) + G'(1) - (G'(1))^2}$.

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Proof.

$$G'(z)\Big|_{z=1} = \sum_{k=0}^{\infty} (z^k)'\mathbb{P}(X = k)\Big|_{z=1} = \sum_{k=0}^{\infty} kz^{k-1}\mathbb{P}(X = k)\Big|_{z=1} =$$

$$\sum_{k=1}^{\infty} kz^{k-1}\mathbb{P}(X = k)\Big|_{z=1} = \sum_{k=1}^{\infty} k\mathbb{P}(X = k) = \mathbb{E}(X).$$

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The proof for the formula for $\mathbb{D}(X)$ is similar.

Recovering the original distribution

The original distribution can be entirely computed from the generating function.

Lemma

$$\mathbb{P}(X = k) = \frac{G^{(k)}(z)\big|_{z=0}}{k!},$$

where $G^{(k)}$ denotes the k-th derivative of G.

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where $G^{(k)}$ denotes the k-th derivative of G.

Specifically,

$$\mathbb{P}(X = 0) = G(0),$$
$$\mathbb{P}(X = 1) = G'(0),$$
$$\mathbb{P}(X = 2) = \frac{G''(0)}{2}$$
$$\text{etc.}$$

Further properties

Let X have generating function $G_X(z)$. What is the generating function of $X+1$?

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$$G_{X+1}(z) = \sum_{k=0}^{\infty} \mathbb{P}(X+1=k)z^k = z\sum_{k=1}^{\infty} \mathbb{P}(X=k-1)z^{k-1}$$
$$= z\sum_{k=0}^{\infty} \mathbb{P}(X=k-1)z^{k-1} = zG_X(z).$$

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$$G_{2X}(z) = \sum_{k=0}^{\infty} \mathbb{P}(2X = k)z^k = \sum_{k=0}^{\infty} \mathbb{P}(X = k/2)(z^2)^{k/2} =$$

$$\sum_{k=0,k \text{ even}}^{\infty} \mathbb{P}(X = k/2)(z^2)^{k/2} =$$

$$\stackrel{l=k/2}{=} \sum_{l=0}^{\infty} \mathbb{P}(X = l)(z^2)^l = G_X(z^2).$$

Sum of two independent random variables

Theorem

If X and Y are independent discrete random variables with generating function $G_X(z)$ and $G_Y(z)$ respectively, then

$$G_{X+Y}(z) = G_X(z)G_Y(z).$$

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Proof (sketch, through example). Let X and Y be the result of rolling two fair 6-sided dice. Let's compute $\mathbb{P}(X+Y=4)$.

$$\mathbb{P}(X+Y=4) =$$
$$\mathbb{P}(X=1, Y=3) + \mathbb{P}(X=2, Y=2) + \mathbb{P}(X=3, Y=1) =$$
$$\frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{3}{36}.$$

Sum of two independent random variables

Let's also compute the coefficient of z^4 in $G_X(z)G_Y(z)$:

$$G_X(z)G_Y(z) = \left(\frac{1}{6}z^1 + \frac{1}{6}z^2 + \cdots + \frac{1}{6}z^6\right)\left(\frac{1}{6}z^1 + \frac{1}{6}z^2 + \cdots + \frac{1}{6}z^6\right).$$

We can get z^4 by taking z^1 from the first bracket and z^3 from the second bracket, or $z^2 \cdot z^2$, or $z^3 \cdot z^1$. The coefficient of these terms gives

$$\frac{1}{6}z^1 \cdot \frac{1}{6}z^3 + \frac{1}{6}z^2 \cdot \frac{1}{6}z^2 + \frac{1}{6}z^3 \cdot \frac{1}{6}z^1 = \frac{3}{36}z^4.$$

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The result of the multiplication of polynomials gives exactly the probability we need.

Deterministic numbers

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For example,

$$G_{X+1}(z) = zG_X(z)$$

follows directly from the previous lemma.

Example - Binomial distribution

Let $X \sim \mathrm{BIN}(n, p)$. Compute its generating function $G(z)$.

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X is the number of successes from n independent trials, so by adding 1 for each success and 0 for each failure, X can be written as

$$X = Y_1 + \cdots + Y_n,$$

where $Y_1, \dots, Y_n$ are iid Bernoulli variables with generating function $G_Y(z) = (1-p) + pz$.

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where $Y_1, \ldots, Y_n$ are iid Bernoulli variables with generating function $G_Y(z) = (1 - p) + pz$. So

$$G(z) = ((1 - p) + pz)^n.$$

Example - Binomial distribution

On the other hand, $G(z)$ can be computed directly from the distribution of X:

$$G(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i.$$

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$$G(z) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i.$$

Due to the binomial theorem, the two formulas are equivalent.

$$G(z) = ((1-p) + pz)^n = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} z^i.$$

Sum with a random number of terms

Let $X_1, X_2, \ldots$ be independent, identically distributed discrete random variables (iid for short) with common generating function $G_X(z)$, and let N be a discrete random variable with generating function $G_N(z)$, independent from all the X's.

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that is, Y is a sum with a random number of terms. What is the generating function of Y?

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Example. We have a blue and a red die, both are 6-sided and fair. We roll the blue die once, then roll the red die a number of times equal to the result of the blue roll. Finally we add all the results of the red rolls. What are the possible values for the sum of the red rolls?

Sum with a random number of terms

Let N be the value of the blue die roll, and $X_1, X_2, \dots$ be the values of the red rolls. Then $Y = X_1 + \cdots + X_N$ is the sum of the red rolls.

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If $N = 1$, then $Y = X_1$, so it has generating function $G_X(z) = \frac{1}{6}z^1 + \cdots + \frac{1}{6}z^6$.

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If $N = 2$, then $Y = X_1 + X_2$, so it has generating function $(G_X(z))^2$.

If $N = 3$, then $Y = X_1 + X_2 + X_3$, so it has generating function $(G_X(z))^3$ etc.

Sum with a random number of terms

Altogether, we have

$$G_Y(z) = \mathbb{E}(z^Y) = \sum_{n=1}^{6} \mathbb{E}(z^Y | N = n)\mathbb{P}(N = n) =$$

$$\frac{1}{6} G_X(z) + \frac{1}{6}(G_X(z))^2 + \cdots + \frac{1}{6}(G_X(z))^6.$$

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Theorem

If $X_1, X_2, \ldots$ are iid discrete random variables with common generating function $G_X(z)$, and N is a discrete random variable with generating function $G_N(z)$, independent from all the X's, and $Y = X_1 + \cdots + X_N$, then

$$G_Y(z) = G_N(G_X(z)).$$

Example

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First, you will have a random number of children.

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First, you will have a random number of children.

Then each of your children will have a random number of children.

Summary

$$G(z) := \sum_{k=0}^{\infty} \mathbb{P}(X = k) z^k.$$

$$G(1) = 1, \quad G(0) = \mathbb{P}(X = 0), \quad \mathbb{P}(X = k) = \frac{G^{(k)}(0)}{k!}$$

$$\mathbb{E}(X) = G'(1), \mathbb{D}(X) = \sqrt{G''(1) + G'(1) - (G'(1))^2}$$

$$G_{X+Y}(z) = G_X(z) G_Y(z) \quad \text{for } X, Y \text{ independent}$$

$$G_{X_1+\cdots+X_N}(z) = G_N(G_X(z)) \quad \text{for } X_1, X_2, \ldots \text{ iid and } N \text{ indep.}$$

The theorem of total expectation applies for the generating function.

Incomplete generating functions

Sometimes we work with discrete random variables which may also take infinity as a value with positive probability. In this case,

$$\sum_{k=0}^{\infty} \mathbb{P}(X = k) = 1 - \mathbb{P}(X = \infty) \leq 1.$$

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$$\sum_{k=0}^{\infty} \mathbb{P}(X = k) = 1 - \mathbb{P}(X = \infty) \leq 1.$$

For such variables, the definition of the generating function

$$G(z) = \sum_{k=0}^{\infty} \mathbb{P}(X = k) z^k$$

still makes sense, but instead of $G(1) = 1$, we now have

$$G(1) \leq 1,$$

with

$$\mathbb{P}(X = \infty) = 1 - G(1).$$

Problem 1

A nonnegative discrete random variable has generating function

$$G(z) = \frac{3}{8} + \frac{3}{8}z + \frac{1}{8}z^2 + \frac{1}{8}z^3.$$

Determine the distribution of X (that is, the $\mathbb{P}(X = k)$ probabilities for $k = 0, 1, 2, \dots$). Calculate its mean and variance as well.

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Solution.

$$\mathbb{P}(X = 0) = \frac{3}{8}, \qquad \mathbb{P}(X = 1) = \frac{3}{8},$$
$$\mathbb{P}(X = 2) = \frac{1}{8}, \qquad \mathbb{P}(X = 3) = \frac{1}{8},$$
$$\mathbb{P}(X = 4) = \mathbb{P}(X = 5) = \cdots = 0.$$

Problem 1

$$G'(z) = \frac{3}{8} + \frac{2}{8}z + \frac{3}{8}z^2,$$
$$G''(z) = \frac{2}{8} + \frac{6}{8}z$$

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and

$$\mathbb{E}(X) = G'(1) = 1,$$
$$\mathrm{Var}(X) = G''(1) + G'(1) - (G'(1))^2 = 1 + 1 - 1^2 = 1.$$

Problem 2

Alice sends a letter to Bob. Postal service is not very reliable; each day, the postman will take the letter to the logistics center with probability $1/3$ (regardless of the past). Once the letter is in the logistics center, each day it is processed with probability $1/5$ (regardless of the past). Once it is processed, shipping it takes 1 day. (So at best, the total delivery time is 1 day.) Let X denote the total delivery time in days. Calculate the generating function and the mean of X.

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Problem 2

Solution. We can write X as

$$X = X_1 + X_2 + X_3,$$

where

- X_1 is the number of days it takes for the postman to take the letter to the logistics center,
- X_2 is the number of days it takes for the logistics center to process the letter, and
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X_1, X_2 and X_3 are independent. What are their distributions?

Problem 2

Then

$$\begin{aligned} X_1 &\sim \mathrm{PGEO}(1/3), & G_1(z) &= \frac{1}{3-2z}, & \mathbb{E}(X_1) &= 2, \\ X_2 &\sim \mathrm{PGEO}(1/5), & G_2(z) &= \frac{1}{5-4z}, & \mathbb{E}(X_2) &= 4, \\ X_3 &= 1, & G_3(z) &= z, & \mathbb{E}(X_3) &= 1. \end{aligned}$$

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(The information that "at best, the total delivery time is 1 day" implies that the geometric distributions are pessimistic.)

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(The information that “at best, the total delivery time is 1 day” implies that the geometric distributions are pessimistic.)

So

$$G(z) = G_1(z)G_2(z)G_3(z) = \frac{1}{3-2z} \cdot \frac{1}{5-4z} \cdot z,$$
$$\mathbb{E}(X) = G'(1) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 2+4+1 = 7.$$

Problem 3

An exam has two parts, A and B. Part B of the exam may be taken only by students who pass part A. Each student passes part A with probability 0.6, independent of the others. Each student who passed part A then passes part B with probability 0.5, independent of the others. 100 students take this test. Let X denote the number of students who pass part A, and Y denote the number of students who pass part B. What is the distribution of X? Calculate G_X, the generating function of X, then derive G_Y, the generating function of Y using G_X. Can we tell the distribution of Y from G_Y?

Problem 3

Solution. Y can be written as

$$Y = Z_1 + \cdots + Z_X,$$

where $Z_1, \ldots, Z_X$ correspond to students who passed part A, and Z_i is 1 if student i has passed part B and 0 if not.

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$$G_Z(z) = 0.5 + 0.5z.$$

$X \sim \mathrm{BIN}(100, 0.6)$, so

$$G_X(z) = (0.4 + 0.6z)^{100}, \qquad \text{and}$$
$$G_Y(z) = G_X(G_Z(z)) = (0.4 + 0.6(0.5 + 0.5z))^{100} = (0.7 + 0.3z)^{100},$$

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Solution. Y can be written as

$$Y = Z_1 + \cdots + Z_X,$$

where $Z_1, \ldots, Z_X$ correspond to students who passed part A, and Z_i is 1 if student i has passed part B and 0 if not. The Z_i's are iid with common generating function

$$G_Z(z) = 0.5 + 0.5z.$$

$X \sim \mathrm{BIN}(100, 0.6)$, so

$$G_X(z) = (0.4 + 0.6z)^{100}, \qquad \text{and}$$
$$G_Y(z) = G_X(G_Z(z)) = (0.4 + 0.6(0.5 + 0.5z))^{100} = (0.7 + 0.3z)^{100},$$

which is the generating function of BIN$(100, 0.3)$.

Problem 3

Another way to see $Y \sim \mathrm{BIN}(100, 0.3)$ is to use total probability.

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Another way to see $Y \sim \mathrm{BIN}(100, 0.3)$ is to use total probability. For each student, the probability that he passes part B can be calculated as

$$
\begin{aligned}
\mathbb{P}(\text{passes B}) = {} & \mathbb{P}(\text{passes B}|\text{passes A})\mathbb{P}(\text{passes A}) + \\
& + \mathbb{P}(\text{passes B}|\text{does not pass A})\mathbb{P}(\text{does not pass A}) = \\
= {} & 0.5 \cdot 0.6 + 0 \cdot 0.4 = 0.3,
\end{aligned}
$$

so each of the 100 students will pass part B with probability 0.3 independently.

Problem 3

The result can be stated as a lemma in general.

Lemma

If $N \sim BIN(n, p)$ *and* $X_1, X_2, \ldots$ *are iid with Bernoulli distribution with parameter* q*, then*

$$Y = X_1 + \cdots + X_N$$

has distribution $BIN(n, pq)$*.*

Problem 4

We roll a fair six-sided die until we get two consecutive sixes. Let Y denote the number of rolls needed to get two consecutive sixes (so for the sequence 1462655661, $Y = 9$). Calculate the generating function of Y, then the mean of Y. Hint: the theorem of total expectation applies for the generating function.

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Solution. Let X_1 denote the number of rolls needed to get the first six.

$X_1 \sim \mathrm{GEO}(1/6)$, so the generating function of X is

$$G_1(z) = \sum_{k=1}^{\infty} \frac{1}{6} \left(\frac{5}{6}\right)^{k-1} z^k = \frac{\frac{1}{6}z}{1 - \frac{5}{6}z} = \frac{z}{6 - 5z}.$$

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If the first roll is not a 6 (which has probability $5/6$), then we rolled once, and the remaining number of rolls needed has the same distribution as the original X_1.

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The corresponding total expectation formula for $G_1(z)$ is

$$G_1(z) = \frac{1}{6}z + \frac{5}{6}zG_1(z),$$

which we can solve to get

$$G_1(z) = \frac{z}{6-5z}$$

again.

Problem 4

Consider Y next, the number of rolls needed to get two consecutive sixes. Since we need at least one six first, Y can be written as

$$Y = X_1 + X_2,$$

where X_1 is again the number of rolls needed to get the first six, and X_2 is the number of rolls needed after the first six. X_1 and X_2 are independent, so

$$G_Y(z) = G_1(z)G_2(z).$$

Problem 4

We apply total expectation to $G_Y(z)$ according to the value of *the roll following the first 6 roll*. If it is also a 6, we are finished: $X_2 = 1$ and $G_Y(z) = G_1(z)z$. This has probability $1/6$.

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$$G_Y(z) = \mathbb{E}(z^Y) = G_1(z)\left(\frac{1}{6}z + \frac{5}{6}zG_Y(z)\right),$$

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$$G_Y(z) = \mathbb{E}(z^Y) = G_1(z)\left(\frac{1}{6}z + \frac{5}{6}zG_Y(z)\right),$$

which is a linear equation for $G_Y(z)$ with the solution

$$G_Y(z) = \frac{z^2}{5z^2 + 30z - 36}, \qquad \text{and} \qquad G_Y'(1) = 42.$$